FIELD BEHAVIOUR NEAR A DIELECTRIC EDGE

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ABSTRACT

A review of the theory of field behaviour near a dielectric edge is presented. An overview of the static results is given and it is shown that the series of Meixner for the dynamic case in general is non-existing. Numerical results for scattering by a dielectric cylinder of square-section indicate that the static results give an accurate account for the rate of growth of the fields in the singular cases, but some disagreement in non-singular cases.

Introduction

The detailed structure of electromagnetic fields near a geometrical configuration is usually not of primary interest, it is often derived quantities like impedance, and scattered fields which are the important ones. However, in numerical calculations of these parameters the convergence of the solution may be improved if a known field behaviour is included in the formulation of the problem, e.g. the well-known singularity near a metallic edge. There may also be high power situations where knowledge of field strength is needed in order to avoid breakdown. It is well-known that a general solution for the penetrable wedge in the time-varying case is not available. For the static case the situation is quite different, as early as 1938 Greenberg [1] gave a solution for the Greens function for the general sectorial medium, i.e. a medium consisting of sectors of different materials, including perfect conductors. The same problem was later (1954), apparently independently solved by Karp[2]. Since all the needed information about the field behaviour may be obtained from the static solution, we shall review this case briefly in the following section and discuss the results.

Static modes on a dielectric wedge

By a static mode we mean a source independent field configuration in the static case, and as shown by Greenberg [1] and Karp [2] the complete solution for a given source position may be found as a superposition of such static modes. The geometry and coordinate system is shown in Fig. 1. It is necessary to distinguish between two different symmetries, one (Fig. 1a) where an electric wall bisects the wedge, such that the electric field is forced to be normal to the plane of symmetry, and one (Fig. 1b) where a magnetic wall bisects the wedge, such that the electric field lies in the plane of symmetry.

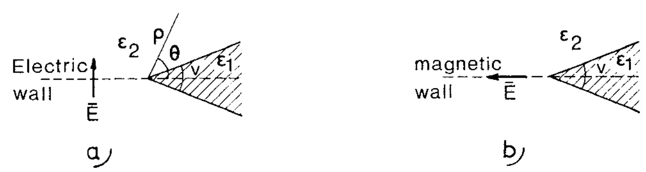

Figure 1: Geometrical configuration and coordinate system.
a) Wedge is bisected by electric wall, $\overline{E}_{tan} = 0$.
b) Wedge is bisected by magnetic wall, $\overline{E}_n = 0$.

Considering case a) first the potential u should satisfy Laplace's equation, u should be continuous across the boundary $\theta = \frac{v}{2}$, $\varepsilon\frac{\partial u}{\partial\theta}$ should be continuous across the same boundary, and u = 0 for = 0 and π.

A solution in each sector is

$$u = A\rho^t \sin t\theta \qquad 0 \leq \theta \leq \frac{v}{2} \tag{1}$$

$$u = B\rho^t \sin t(\pi - \theta) \qquad \frac{v}{2} \leq \theta \leq \pi \tag{2}$$

Application of the boundary conditions yields

$$\varepsilon_1 \cos t\frac{v}{2}\, \sin t(\pi - \frac{v}{2}) = -\varepsilon_2 \sin t\frac{v}{2} \cos t(\pi - \frac{v}{2}) \tag{3}$$

which may be transformed to the following equation for the determination of t for the static modes,

$$D_e(t) = \sin t\pi - a \sin t(v - \pi) = 0 \tag{4}$$

where

$$a = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

In case b) the only difference is that $\frac{\partial u}{\partial\theta} = 0$ for $\theta = 0$ and π and the following transcendental equation is obtained

$$D_m(t) = \sin t\pi + a \sin t(v - \pi) = 0 \tag{5}$$

Eqs. 4 and 5 have an infinity of solutions, but since the potential must remain finite only positive values of t are allowed. The field strengths, E_ρ and E_θ, have a ρ dependence of ρ^{t-1}, thus these transverse fields tend to infinity or to zero, depending on whether the smallest positive value of t, determined from (4) or (5), is less than or greater than 1, respectively.

A completely analogous situation with a magnetic wedge may be treated, but the results may be found simply by interchanging ε with μ, E with H and electric wall with magnetic wall, so no further discussion of this case will be given.

It is worth stressing that only the fields transverse to the edge may be singular. A component of electric field parallel to the edge does not exhibit the above mentioned behaviour, thus E_z may tend to a constant when ρ tends to zero.

Numerical results for t found from eqs. (4) and (5) are given by Meixner [3] and by Bobrovnikov and Zamaraeva [4], who also treat some cases with several dielectrics.

Apart from the actual numbers it is of interest to get an overview over the field behaviour in various configurations. This is attempted in Fig. 2, where the dielectric constant in the shaded area is assumed to be the largest. The symmetry is indicated by the electric field vector as in Fig. 1. For comparison the results for a perfectly conducting wedge are also indicated. By consulting the figure one may easily get an idea of the field behaviour near the edge, the precise values of t must be obtained from eqs. 4 and 5.

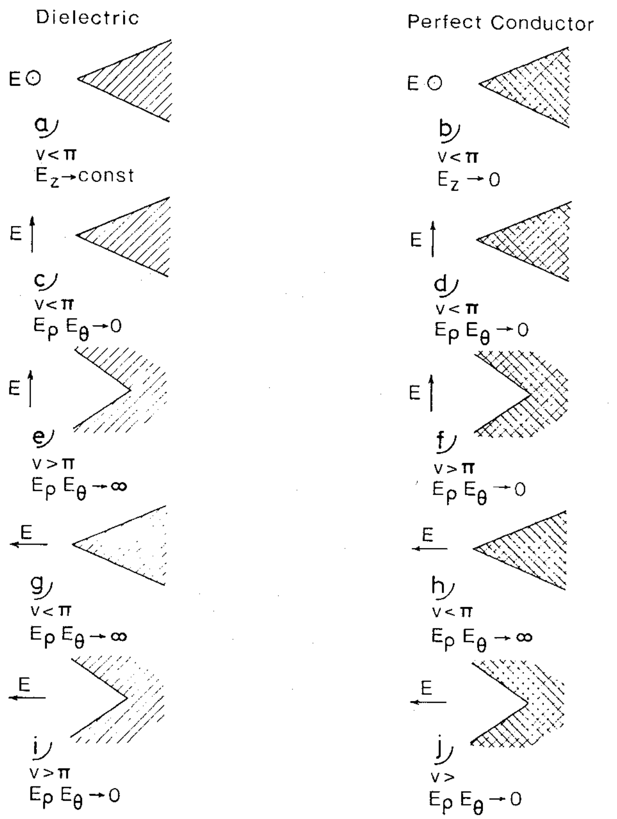

Figure 2: Schematic presentation of field behaviour near the tip of a dielectric and metallic wedge. In the dielectric case the shaded area has the highest dielectric constant. The limiting values indicate the proces of letting ϱ tend to zero.

It is of interest to compare the two columns of Fig. 2. Often the conducting case may be found as a limiting case for the dielectric when $\varepsilon_1 \to \infty$, there are however exceptions. Comparing 2e with 2f, the electric fields are singular in the dielectric case, but not in the metallic case. The reason is the following. Letting $\varepsilon_1 \to \infty$ in (3) we find two solutions

i) $\cos t \frac{v}{2} = 0$ (6)

The first root is $t_1^i = \frac{\pi}{v} < 1$ for $v > \pi$

ii) $\sin t(\pi - \frac{v}{2}) = 0$ (7)

The first root is $t_1^{ii} = \frac{2\pi}{2\pi - v} > 1$

Case (ii) gives A = 0, i.e. zero fields in the dielectric, and the solution is compatible with the metallic case. The singular case, case (i), requires non-vanishing fields in both media, since

$$A = -\cos t_1 \pi \cdot B, \tag{8}$$

and must therefore be ruled out in the metallic case. However, in the dielectric case we have a singular situation, and assuming that a wedge of finite conductivity behaves like a dielectric wedge, we may infer that in the corners of a rectangular waveguide the electric fields may be infinite. Comparing 2h and 2g we see that the transverse electric fields will tend to infinity even for finite conductivity.

Meixner's theory for the time-varying case

The electromagnetic case has been studied by Meixner [3], who suggested that the fields in each medium could be expanded as follows

$$E_\varrho = a_0 \varrho^{t-1} + a_1 \varrho^t + a_2 \varrho^{t+1} + \dots \tag{9}$$

and similarly for the other components, E_θ and H_z.

Matching boundary conditions term by term one gets an infinite, recursive set of equations, where the first set of equations reproduces the static results of Greenberg and Karp. When one tries to construct further terms in (9), it turns out that an inhomogeneous set of equations with determinant $D_e(t+2N)$ results, where t is given by (4). This determinant is zero when

$$v = \frac{K}{N}\pi \tag{10}$$

where K and N are integers. Thus Meixner's solution does not exist for an infinity of wedge angles, including $v = \frac{\pi}{2}$. This does not necessarily mean that the first term (the static term) is not part of a complete solution, but eq. 9 cannot be correct. If the wedge is perfectly conducting the problems mentioned do not occur and eq. 9 is correct.

Numerical results for a dielectric cylinder of rectangular cross-section

In order to study an actual case in detail some numerical computations have been performed. Scattering of electromagnetic waves by a square cylinder of relative dielectric constant of 10 has been computed by integral equation techniques. Radius of curvature

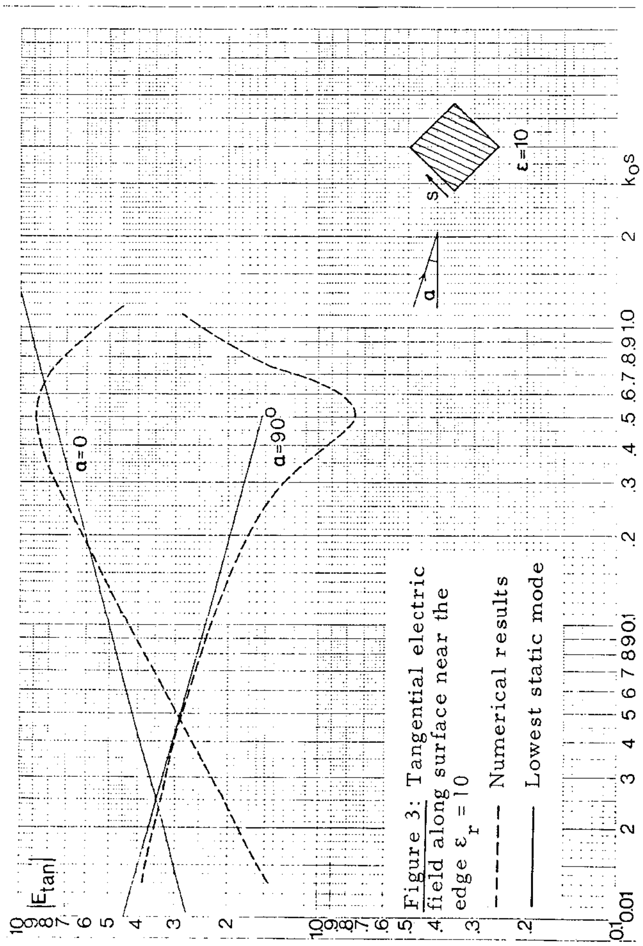

Figure 3: Tangential electric field along surface near the edge $\varepsilon_r = 10$
- - - - Numerical results
——— Lowest static mode

$(k_o r)$ at the corners equals 0.01 and the side length $(k_o d)$ equals 3.5. The direction of incidence is normal to the cylinder axis and the H-component of the incident wave is parallel to the cylinder axis. In Fig. 2 the electric field parallel to the surface is plotted on a log-log scale as a function of distance along the side. This representation gives a straight line for the static modes of eqs. 1 and 2.

We note first that the field tends to zero in the case $\alpha = 0$ in agreement with Fig. 2c and tends to infinity for $\alpha = 90^o$ in agreement with Fig. 2g. In the singular case the agreement with the slope of the static wedge mode is good, while there is a definite disagreement in the non-singular case. The disagreement is unexplained at present but may be related to the failure of Meixner's series as indicated above.

References

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